

May, 1998

USITP-98-06

A SPINNING ANTI-DE SITTER WORMHOLE

Stefan Åminneborg^{*1}

Ingemar Bengtsson^{**2}

Sören Holst^{***3}

** Norra Reals Gymnasium, S-113 55 Stockholm, Sweden*

*** Fysikum, Stockholm University, Box 6730, S-113 85 Stockholm, Sweden*

Abstract

We construct a 2+1 dimensional spacetime of constant curvature whose spatial topology is that of a torus with one asymptotic region attached. It is also a black hole whose event horizon spins with respect to infinity. An observer entering the hole necessarily ends up at a "singularity"; there are no inner horizons. In the construction we take the quotient of 2+1 dimensional anti-de Sitter space by a discrete group Γ . A key part of the analysis proceeds by studying the action of Γ on the boundary of the spacetime.

¹Email address: stefan@vanosf.physto.se

²Email address: ingemar@vana.physto.se

³Email address: holst@vanosf.physto.se

1. Introduction

A topological geon is a gravitating object in which a non-trivial spatial topology is somehow "localized" in the interior. Geons have been much discussed over the years [1]. By now a number of general results are available [2] but exact solutions remain hard to come by. In a previous publication (with Brill and Peldán [3]) we pointed out that if we restrict ourselves to 2+1 dimensional spacetimes then wormhole solutions of the vacuum equations are easy to construct, provided that a negative cosmological constant is admitted. Our definition of a wormhole is that it is a solution of Einstein's equations whose spatial topology is that of a non-simply connected surface with one asymptotic region attached. We constructed all such solutions under the restriction that they be time symmetric, and we analyzed how the general results apply to these spacetimes. The reason why these solutions are so easy to obtain is that in 2+1 dimensions all solutions of the vacuum equations have constant curvature; since we assumed a negative cosmological constant they are locally isometric to 2+1 dimensional anti-de Sitter space. It is believed [4] (and we assume it to be true, also in the non-compact case) that all such solutions can be obtained as the quotient space adS/Γ , where adS denotes an open region of 2+1 dimensional anti-de Sitter space and Γ is some discrete subgroup of isometries of this spacetime. Thus the question becomes how to identify those choices of discrete subgroups of isometries that lead to quotient spaces having the wormhole topology. An additional requirement is that all the "singularities" that occur to the future of some smooth spacelike slice must be hidden from view at infinity by an event horizon.

We are faced with a Lorentzian version of Clifford-Klein's problem [5]: To determine all three dimensional spacetimes of constant negative curvature which are everywhere regular in the sense specified.

We find this problem interesting in itself, and also because the resulting solutions can be used to test ideas about (say) topological censorship [6] or black hole entropy [7]. Unfortunately the present paper rests to a large extent on our previous work, and we found it difficult to make it self-contained while keeping its length within reasonable limits. For this reason section 2 provides a somewhat extended introduction; the organization of the rest of the paper is given away at the end of that section. Anyway, the key problem to be solved here is the construction of a wormhole whose event horizon spins with respect to infinity.

To get the priorities clear, it remains to say that our papers are basically an elaboration of results due to Bañados, Henneaux, Teitelboim and Zanelli [8].

2. The Introduction Continued

The isometry group of 2+1 dimensional anti-de Sitter space is $SO(2, 2)$, a group that obeys the local isomorphism

$$SO(2, 2) \sim SL(2, \mathbf{R}) \otimes SL(2, \mathbf{R}) . \quad (1)$$

The boundary of conformally compactified anti-de Sitter space is a 1+1 dimensional cylinder whose conformal structure is fixed, and the isometry group of the interior acts as the conformal group of the boundary. We refer to the boundary as \mathcal{J} ("script I"). To construct our wormholes we first select a discrete subgroup Γ consisting of elements that can be reached by exponentiating some Killing vectors. The subgroup will be defined by listing a finite number of generators of Γ . As our covering space we choose an open region of anti-de Sitter space which is such that the flows of all the Killing vectors that generate elements of Γ are spacelike within the region. This ensures that there are no closed timelike curves in the quotient space. The boundary of covering space is considered to give rise to "singularities" in the quotient space (and we use quotation marks since it may to some extent be possible to analytically extend through the "singularities" at the cost of admitting closed timelike curves—this notion of "singularity" has been fully discussed in earlier papers [8], to which the sceptical reader is referred). The causal structure of the quotient space is most conveniently analyzed directly in covering space [9]. Now a fully explicit description of the covering space needs a fully explicit description of all the infinite elements of Γ , and in general this is very hard to obtain. However, a description that is sufficiently explicit for our purposes needs only a sufficient amount of control over the location of those fixed points of the group that do occur on \mathcal{J} .

In our previous publication [3] we assumed time symmetry, or in other words that Γ is a subgroup of a diagonal $SL(2, \mathbf{R})$ subgroup that transforms a particular spacelike slice onto itself. This slice is a Poincaré disk, and the action of the diagonal subgroup is that of Möbius transformations preserving the disk. It is then possible to solve the problem in two steps [10]: The group Γ transforms the Poincaré disk onto itself and therefore one can begin by choosing Γ so that it gives the appropriate smooth spatial topology to this surface. Once this has been ensured the action of Γ is extended to a suitable open region of the full anti-de Sitter space and the resulting spacetime analyzed in detail. In the first step we have to ensure that none of the infinite set of elements of Γ have fixed points within the disk. In group theoretical terms, this requirement translates into the statement that all the elements of Γ have to belong to the hyperbolic conjugacy class. Hyperbolic elements always have a pair of fixed points lying on the boundary of the disk, but this is allowed. In practice Γ will be defined by choosing a fundamental region that tessellates the disk. By specifying which

pairs of edges of the fundamental region that are to be identified one specifies a finite set of generators of the group Γ , and the condition on the rest of the elements is automatically met provided that the manifold that one obtains when gluing the edges of the fundamental region together is smooth. Fixed points do occur on the boundary of the disk, and it is not too difficult to pin them down with the accuracy that one needs. (Details will be given later.) Thus at the moment of time symmetry the covering space of the interior is the entire disk, and the covering space of the boundary is all of the boundary except for the fixed points. In the next step of the construction we must first define the boundaries of the covering space of our spacetime. According to our definition [9] these boundaries are the "singularity surfaces" where a Killing vector generating some element of Γ becomes lightlike. When Γ belongs to the diagonal subgroup of $SO(2, 2)$ it happens that the "singularity surfaces" are null planes, that is to say light cones with their vertices on \mathcal{J} . These vertices are precisely the fixed points that we encountered in the first step. In this way we can understand our covering space. Additional fixed points occur where the null planes intersect to the future (and to the past) of our initial data slice. This is where spacetime, and \mathcal{J} , ends. Only a part of spacetime can be seen from infinity, and in this sense our wormhole turns out to be a *bona fide* black hole. The event horizon is the backwards light cone of the last point on \mathcal{J} . (Again, details will be given later.)

Now time symmetry is a strong restriction that we want to lift. The restriction was imposed only because it facilitates finding the location of some of the fixed points on \mathcal{J} . However, the same information can be found in a different way. The point—easy to see for those readers who are familiar with conformal field theory—is that if we use light cone coordinates u and v on \mathcal{J} , then the action of the two $SL(2, \mathbf{R})$ factors of $SO(2, 2)$ decouples. One factor will give projective transformations on the lines defined by constant v , and the other on the lines defined by constant u . This observation is enough to enable us to formulate conditions on the generators of Γ so that the quotient of any region of \mathcal{J} where the Killing vectors are spacelike becomes a smooth manifold terminated by a "singularity" caused by fixed points. In other words, in the time symmetric case we were able to start off our discussion by first making a sufficiently accurate picture of the covering space of the initial data slice, but we now see that we can instead start from a sufficiently accurate picture of the covering space of \mathcal{J} . In the next step the analysis proceeds as in the time symmetric case, although there are some differences. For one thing the event horizon now turns out to "spin" relative to \mathcal{J} . There is also a definite difficulty caused by the fact that the "singularity surfaces" are no longer null planes, even though their intersections with \mathcal{J} are null [9]. In the case of the Kerr-like BTZ black hole [8] what happens is that the covering space increases as the spin of

the event horizon increases, with the result that a "mouth" opens up in the interior through which it is possible to travel to another universe. Our analysis shows that this is not the case for the spinning wormhole. The interior of a wormhole always ends in a "singularity", and there are no inner horizons.

There is a rather large amount of formulæ in the body of the paper. This is because we decided to include enough detail so that we could sketch an explicit calculation of the angular velocity of the horizon in terms of the parameters characterizing the generators of Γ . It seemed worthwhile to do this even though it would be possible to understand the construction with less attention to detail.

After this outline of the argument, we are ready to state the contents of our paper: Section 3 provides the necessary background information about anti-de Sitter space and its isometry group. Section 4 gives an account of the BTZ black holes [8]. Although this is standard material our presentation is new, and it is intended to show how information may be extracted from \mathcal{J} . In section 5 we revisit the spinless wormholes [3], but again from a new point of view that stresses the properties of the covering space. In section 6 we finally prove that a spinning wormhole exists, we compute the angular velocity of its horizon, and we show that its domain of exterior communication is isometric to that of a BTZ black hole. We also count the number of parameters in our solution. In section 7 we discuss the interior of the spinning wormhole and prove that it ends in a "singularity"; there is only one asymptotic region and—according to an admittedly somewhat heuristic argument—there are no inner horizons. Finally, section 8 states our conclusions and provides a list of open questions.

Although we have made an effort to write a readable paper it will help if the reader has at least a nodding acquaintance with our previous publications [3] [9], as well as with standard definitions in black hole physics [11].

3. 2+1 anti-de Sitter space and its isometry group.

Three dimensional anti-de Sitter space can be defined as the quadric surface

$$X^2 + Y^2 - U^2 - V^2 = -1 \quad (2)$$

embedded in a flat space with the metric

$$ds^2 = dX^2 + dY^2 - dU^2 - dV^2 . \quad (3)$$

A description of anti-de Sitter space in terms of embedding coordinates is not quite enough for our purposes since we will want to define the conformal boundary of spacetime. Therefore we introduce the intrinsic coordinates t, ρ, ϕ through

$$X = \frac{2\rho}{1 - \rho^2} \cos \phi \quad Y = \frac{2\rho}{1 - \rho^2} \sin \phi \quad (4)$$

$$U = \frac{1 + \rho^2}{1 - \rho^2} \cos t \quad V = \frac{1 + \rho^2}{1 - \rho^2} \sin t. \quad (5)$$

Then (the universal covering space of) anti-de Sitter space becomes simply the interior of the (infinite) cylinder $\rho < 1$, and its conformal compactification is obtained by adjoining the surface of the cylinder $\rho = 1$. This boundary is \mathcal{J} , the set of endpoints of all lightlike geodesics. Note that past \mathcal{J} , future \mathcal{J} , and spacelike infinity all coincide for anti-de Sitter space. The asymptotic structure is therefore quite different from that of Minkowski space (and it is this very feature that enables one to find black holes with constant curvature). The metric on anti-de Sitter space becomes in our coordinates

$$ds^2 = - \left(\frac{1 + \rho^2}{1 - \rho^2} \right)^2 dt^2 + \frac{4}{(1 - \rho^2)^2} (d\rho^2 + \rho^2 d\phi^2). \quad (6)$$

At constant t this is the metric for the hyperbolic plane, also known as the Poincaré disk.

A metric on \mathcal{J} can be obtained by multiplying the metric of anti-de Sitter space with a suitable factor that vanishes on \mathcal{J} . In this way we obtain an "unphysical" metric that extends smoothly to the boundary, and it will induce a metric there. Our choice for the unphysical metric is

$$d\hat{s}^2 = \frac{1}{U^2 + V^2} ds^2 = -dt^2 + \frac{4}{(1 + \rho^2)^2} (d\rho^2 + \rho^2 d\phi^2). \quad (7)$$

This metric induces the flat metric on the cylindrical surface $\rho = 1$. It will prove convenient to introduce light cone coordinates on the boundary;

$$u = t - \phi \quad v = t + \phi. \quad (8)$$

In terms of these coordinates the metric on \mathcal{J} takes the form

$$d\hat{s}^2 = -dudv. \quad (9)$$

For a discussion of the conformal compactification of asymptotically anti-de Sitter spaces in general, see ref. [12].

The group of isometries of three dimensional anti-de Sitter space is the six dimensional group $O(2, 2)$. Its connected component $SO_o(2, 2)$ is a direct product

$$SO_o(2, 2) = \frac{SL(2, \mathbf{R}) \otimes SL(2, \mathbf{R})}{Z^2}. \quad (10)$$

To see this, write the defining equation of the quadric as a condition on the determinant of a matrix:

$$|\mathbf{X}| = \begin{vmatrix} V + X & Y + U \\ Y - U & V - X \end{vmatrix} = 1. \quad (11)$$

This condition is clearly preserved by any transformation of the form

$$\mathbf{X} \rightarrow \mathbf{X}' = g\mathbf{X}\tilde{g}^{-1}, \quad g \in SL(2, \mathbf{R}), \quad \tilde{g} \in SL(2, \mathbf{R}). \quad (12)$$

In this way any element G of $SO_0(2, 2)$ may be identified with an equivalence class of two elements in the direct product of two $SL(2, \mathbf{R})$ s;

$$G = (g, \tilde{g}) \sim (-g, -\tilde{g}). \quad (13)$$

Group multiplication is defined in the obvious way. An example of an element of $SO(2, 2)$ that does not lie in the connected component is the reflection

$$\pi : (X, Y, U, V) \rightarrow (X, -Y, U, -V). \quad (14)$$

An example of an element of $O(2, 2)$ that does not lie in $SO(2, 2)$ is the reflection

$$\Pi : (X, Y, U, V) \rightarrow (X, Y, U, -V). \quad (15)$$

Once the structure of $SL(2, \mathbf{R})$ has been understood an understanding of $O(2, 2)$ is not far behind.

The group $SL(2, \mathbf{R})$ is defined as the group formed by all two by two matrices of unit determinant. Let us coordinatize an arbitrary element $g \in SL(2, \mathbf{R})$ as

$$g = \begin{pmatrix} D + A & B + C \\ B - C & D - A \end{pmatrix}, \quad |g| = 1. \quad (16)$$

We see that the group manifold of $SL(2, \mathbf{R})$ is precisely 2+1 dimensional anti-de Sitter space (a happy accident!). Hence we can draw a Penrose diagram of our group, see figure 1. It is naturally divided into regions by the lightcones having vertices at the group elements $\pm e$. If we identify group elements that differ by a sign we obtain $SO_0(2, 1)$, the group of Möbius transformations that preserve the unit circle and map its interior to itself. Alternatively we can describe $SO_0(2, 1)$ as the group of projective transformations of the circle. Both viewpoints will figure prominently below.

We will be interested in understanding the conjugacy classes of the group, and we will want to know which group elements can be reached by exponentiating its Lie algebra elements. Let us begin with the conjugacy classes of $SL(2, \mathbf{R})$, which are equivalence classes of elements that can be connected to each other by means of conjugation with some group element g_0 ;

$$g \sim g' = g_0 g g_0^{-1}. \quad (17)$$

The trace of g is invariant under conjugation. The conjugacy classes are surfaces that foliate the group, and in our coordinates it is easy to see how:

$$\text{Tr } g = 2D \quad \Rightarrow \quad A^2 + B^2 - C^2 = -1 + D^2 = -1 + \left(\frac{\text{Tr } g}{2}\right)^2. \quad (18)$$

In this way we find that the various regions of our Penrose diagram correspond to conjugacy classes according to the following scheme:

$\text{Tr } g > 2$	Region I	(hyperbolic)
$\text{Tr } g < -2$	Region II	(hyperbolic)
$ \text{Tr } g < 2$	Regions III, IV	(elliptic)
$ \text{Tr } g = 2$	The light cones	(parabolic).

The notation "hyperbolic, elliptic and parabolic" refers to the classification of Möbius transformations.

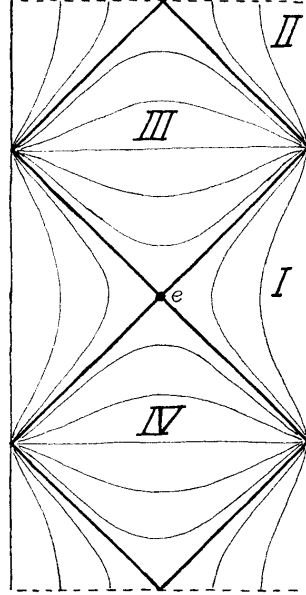


Fig. 1. The Penrose diagram of $SL(2, \mathbf{R})$. The lightcones shown naturally divide it into regions, and these have been foliated by conjugacy classes. Note that region II cannot be reached by exponentiation from the unit element e .

If we exponentiate a Lie algebra element we obtain a geodesic in the group manifold that starts out from the unit element e . From this we may deduce that region II cannot be reached by exponentiation in this way. The exponentiation will be done explicitly in section 6, where we will regard the group as generating projective transformations of the circle.

We now turn to $O(2, 2)$, whose connected component is a product of two

three dimensional anti-de Sitter spaces quotiented by Z^2 . Our interest lies in discrete subgroups of $O(2, 2)$ whose generators can be reached by exponentiation from the identity (this is a restriction that we impose somewhat arbitrarily), and our task is therefore to divide the connected component into conjugacy classes with respect to the full group $O(2, 2)$. Given our knowledge of $SL(2, \mathbf{R})$ we already understand conjugation with elements in the connected component (because it can be carried out separately on the two factors of the group). Conjugation with the reflection Π that was defined above has the effect of exchanging the two factors;

$$\Pi(g_1, g_2)\Pi^{-1} = (g_2, g_1) . \quad (19)$$

With this bit of additional information the problem is solved, and we find the following conjugacy classes (the result and the notation is due to Bañados et al. [8]):

I_a :	elliptic \otimes hyperbolic
I_b :	hyperbolic \otimes hyperbolic
I_c :	elliptic \otimes elliptic
II_a :	parabolic \otimes hyperbolic
II_b :	parabolic \otimes elliptic
III_a :	parabolic \otimes parabolic, future \otimes past
III_b :	parabolic \otimes parabolic, future \otimes future.

As it turns out the only class of interest to us is type I_b . We will be interested in formulating conditions that guarantee that all the elements of a discrete subgroup Γ lie in this class. Why this is the problem will begin to be apparent in the next section.

We end this section with a list of the Killing vectors of anti-de Sitter space. In embedding coordinates they are

$$J_{XY} = X\partial_Y - Y\partial_X \quad J_{XU} = X\partial_U + U\partial_X \quad (20)$$

and so on. Since the Lie algebra of $SO(2, 2)$ splits in a direct sum it is convenient to group them into two mutually commuting sets. We will also need to know how the Killing vectors act on \mathcal{J} ; here is a list that accomplishes both purposes:

$$J_1 \equiv -\frac{1}{2}(J_{XU} + J_{YV}) = \sin u \partial_u \quad \tilde{J}_1 \equiv -\frac{1}{2}(J_{XU} - J_{YV}) = \sin v \partial_v \quad (21)$$

$$J_2 \equiv -\frac{1}{2}(J_{XV} - J_{YU}) = -\cos u \partial_u \quad \tilde{J}_2 \equiv -\frac{1}{2}(J_{XV} + J_{YU}) = -\cos v \partial_v \quad (22)$$

$$J_3 \equiv -\frac{1}{2}(J_{XY} - J_{UV}) = \partial_u \quad \tilde{J}_3 \equiv \frac{1}{2}(J_{XY} + J_{UV}) = \partial_v . \quad (23)$$

Admittedly there is some abuse of notation here—the expressions in terms of embedding coordinates are only valid in the interior, and the expressions in

light cone coordinates only on the boundary—but we hope that the meaning is clear. As promised, the use of light cone coordinates on \mathcal{J} makes the split into two mutually commuting sets manifest. Note that on \mathcal{J} these vector fields are in general only conformal Killing vectors with respect to the metric that we introduced on \mathcal{J} . But then it is only the conformal structure on \mathcal{J} that has an invariant significance. As a side remark we observe that the conformal group in two dimensions is infinite dimensional, and that in fact the entire conformal group acts as an asymptotic symmetry group here [13]. But this fact will play no role in the present paper.

The classification of the group elements into conjugacy classes applies to the Killing vectors as well. The general form of a Killing vector that belongs to the hyperbolic conjugacy class of $SL(2, \mathbf{R})$ is

$$\xi = x_1 J_1 + x_2 J_2 + x_3 J_3 , \quad x_1^2 + x_2^2 - x_3^2 > 0 . \quad (24)$$

Up to normalization we can write this in the alternative form

$$\xi = \sin \alpha J_1 - \cos \alpha J_2 - \cos \beta J_3 , \quad \beta \neq 0 . \quad (25)$$

The fact that an arbitrary arbitrary hyperbolic Killing vector can be written like this happens to be useful later. And with this observation our group theoretical preliminaries are finally at an end.

4. The BTZ Black Hole Viewed From \mathcal{J} .

As shown by Bañados, Henneaux, Teitelboim and Zanelli [8] a black hole solution of Einstein's equations can be obtained by choosing a Killing vector from the conjugacy class I_b , letting it generate a finite group element γ , and taking the spacetime to be adS/Γ , where adS is a region where the flow of the Killing vector is spacelike and Γ is the cyclic group generated by γ . The solution has many properties in common with the Kerr family of black holes in four space-time dimensions. The reason why the conjugacy class I_b is singled out here has to do with the location of the fixed points and the "singularity surfaces" (where the generating Killing vector becomes lightlike); a geometrical analysis of this question, valid in 2+1 and 3+1 dimensions, can be found in the literature [9]. In 2+1 dimensions there is an extremal black hole that results from choosing a Killing vector of type III_a , but it will play no role in the present paper.

We will present the BTZ black hole from a new point of view that will prove useful in the following sections. The construction starts by selecting a group element generated by a Killing vector of type I_b , say

$$\gamma_1 = e^{\xi_1} , \quad \xi_1 = a J_{XU} + b J_{YV} = -(a+b) J_1 - (a-b) \tilde{J}_1 . \quad (26)$$

Without loss of essential generality we may choose $a > b \geq 0$. Let us consider the flow of this Killing vector on \mathcal{J} . The flow will be spacelike in the square defined by (say)

$$-\pi < u < 0 \quad 0 < v < \pi . \quad (27)$$

This region will be the covering space of an asymptotic region of the BTZ black hole. To show that the BTZ black hole is asymptotically anti-de Sitter we introduce a new metric on \mathcal{J} :

$$d\hat{s}^2 = -\frac{dudv}{(a^2 - b^2) \sin u \sin v} . \quad (28)$$

This metric is related by a conformal rescaling to the one introduced above and the transformation is well defined in the entire covering space, so this is allowed. Our Killing vector ξ_1 is indeed a true Killing vector of this new metric. When the identification is carried through, the Killing vector ξ_1 will have closed flow lines and can therefore be regarded as the generator of rotations in the quotient space, which has the topology of a cylinder. The quotient space also admits a global Killing vector orthogonal to the generator of rotations. It has a timelike flow and can therefore be regarded as the generator of time translations. The boundary of the quotient space has now been shown to be a timelike cylinder with the appropriate conformal structure—hence the BTZ black hole is asymptotically anti-de Sitter. To be explicit about it, the generator of asymptotic rotations is

$$\xi_{rot} \equiv \xi_1 = -(a + b) \sin u \partial_u - (a - b) \sin v \partial_v . \quad (29)$$

Up to normalization the generator of time translations is determined by the condition

$$\xi_{rot} \cdot \xi_{time} = 0 . \quad (30)$$

A solution is

$$\xi_{time} = -(a + b) \sin u \partial_u + (a - b) \sin v \partial_v . \quad (31)$$

When evaluated with respect to the new metric on \mathcal{J} the norms of these generators are

$$||\xi_{rot}||_\infty^2 = -||\xi_{time}||_\infty^2 = 1 . \quad (32)$$

It is useful to have a rough idea about the flow of these Killing vectors for various values of the parameter b/a , see figure 2.

What about the spin of the BTZ black hole? To compute it we need to know that the covering space of the event horizon is the backwards light cone of the last point on \mathcal{J} , whose light cone coordinates are

$$(u, v) = (u_P, v_P) = (0, \pi) . \quad (33)$$

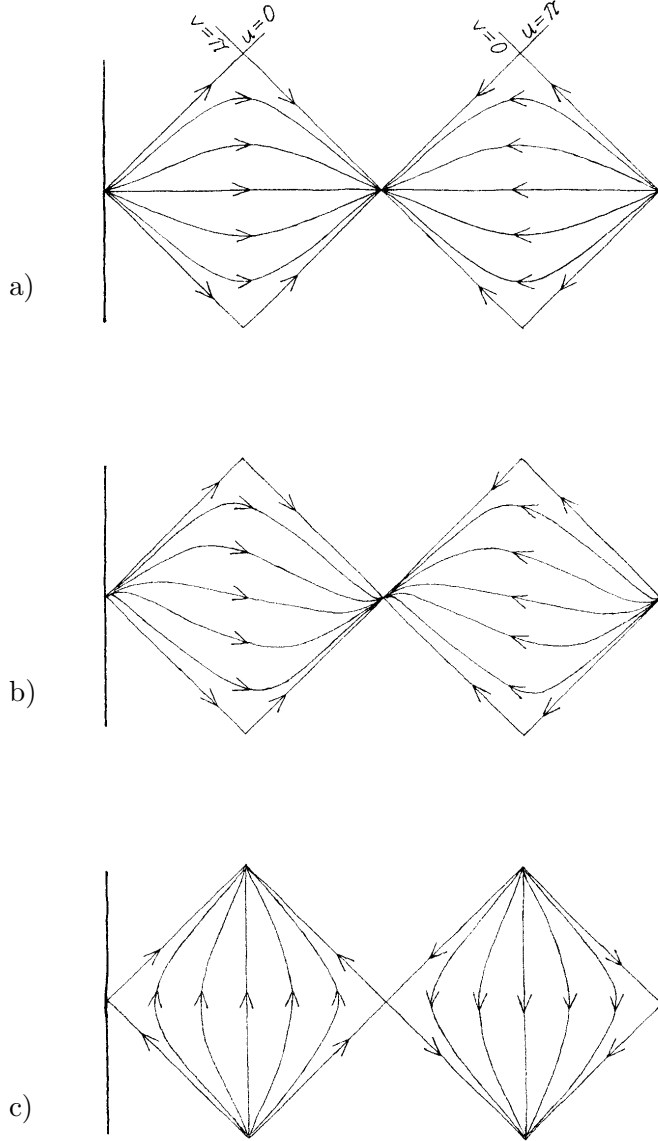


Fig. 2. These pictures all show various Killing flows on the surface of the anti-de Sitter cylinder—that is, on \mathcal{J} —which has been cut along $\phi = 0$ and flattened out. The null coordinates u and v defined in eqn. (8) are indicated in a). The flows are drawn only in the regions belonging to the covering space of the BTZ black hole, that is, where the Killing flow ξ_1 generating the identification is spacelike.

- a) ξ_1 (or ξ_{rot}) for the spinless BTZ hole ($b = 0$).
- b) ξ_1 (or ξ_{rot}) for the spinning BTZ hole: $\Omega \equiv b/a = 1/4$.
- c) The flow ξ_{hor} generating the horizon. This flow is independent of the spin, and it is easy to see from these pictures when the black hole spins (cf. eqn. (37)).

In embedding coordinates a light cone with its vertex on \mathcal{J} , say at $(t, \phi) = (t_P, \phi_P)$, is given by the equation

$$\cos \phi_P X + \sin \phi_P Y - \cos t_P U - \sin t_P V = 0 . \quad (34)$$

The formula holds for arbitrary points on \mathcal{J} . Adapting it to our case we find that the union of the backwards lightcone of the last point on \mathcal{J} with the forwards lightcone of the first point on \mathcal{J} is given by

$$(Y - V)(Y + V) = Y^2 - V^2 = 0 . \quad (35)$$

This is a bifurcate surface and it is lightlike since it contains its own normal; the normal coincides with the Killing vector

$$\xi_{hor} = J_{YV} . \quad (36)$$

Therefore the event horizon lies in the bifurcate Killing horizon ruled by the integral curves of this Killing vector, to which we refer as the horizon generator. Now we can bring the horizon generator back to \mathcal{J} , where it can be written as a linear combination

$$\xi_{hor} = -\sin u \partial_u + \sin v \partial_v = \frac{a}{a^2 - b^2} (\xi_{time} - \Omega \xi_{rot}) , \quad (37)$$

where

$$\Omega = \frac{b}{a} . \quad (38)$$

The overall normalization is not relevant. On the other hand the parameter Ω is a measure of how much rotation the horizon generator contains. By a definition that applies to any stationary black hole Ω is the angular velocity of the event horizon.

Again by definition, the domain of exterior communication of a black hole contains all points that can be reached from \mathcal{J} by both future and past directed causal curves. In our case it is bounded on one side by \mathcal{J} and on the other by the bifurcate Killing horizon that we found. Until section 7 our interest will be entirely confined to the domain of exterior communication for all the black holes that we will construct.

By the way, there is one flow line of ξ_1 that is a spacelike geodesic, namely the line $Y = V = 0$. When the identification is carried through this becomes a closed geodesic of length a . But the intersection of the event horizon (a null plane in covering space) with the Poincaré disk defined by $V = 0$ is also a geodesic, since both surfaces are totally geodesic. It is the same geodesic defined in two different ways; hence the length of the event horizon—which is the entropy up to a numerical factor—is just a . We also observe that the mass and the spin of the black hole can be expressed as functions of a and b , but the

definition really requires the kind of careful attention given to it in ref. [8]. For our purposes the angular velocity is enough.

We will now repeat the BTZ construction for an arbitrary Killing vector belonging to type I_b . This may seem to be a pointless exercise since any such Killing vector can be brought to the form already considered by means of conjugation. We do it because in our discussion of the wormhole we will want to consider generating Killing vectors "in arbitrary position", and then the formulæ that we obtain now will prove useful. So, by the remark at the end of the previous section, on \mathcal{J} an arbitrary type I_b Killing vector can always be written (up to normalization) in the form

$$\begin{aligned} \xi = & -\frac{1}{2}(\sin \alpha \sin u + \cos \alpha \cos u - \cos \beta) \partial_u \\ & + \frac{k}{2}(\sin \tilde{\alpha} \sin v + \cos \tilde{\alpha} \cos v - \cos \tilde{\beta}) \partial_v . \end{aligned} \quad (39)$$

There are five real parameters. We can set

$$\alpha \equiv \frac{u_P + u_{P'}}{2} \quad \beta \equiv \frac{u_P - u_{P'}}{2} \quad \tilde{\alpha} \equiv \frac{v_P + v_{P'}}{2} \quad \tilde{\beta} \equiv \frac{v_P - v_{P'}}{2} , \quad (40)$$

and without loss of essential generality we can take $u_P > u_{P'}$, $v_P > v_{P'}$ and $k > 0$. The apparently somewhat eccentric choice of these parameters is explained when we observe that trigonometric identities can be employed to rewrite the Killing vector in the convenient form

$$\xi = \sin \left(\frac{u - u_P}{2} \right) \sin \left(\frac{u - u_{P'}}{2} \right) \partial_u - k \sin \left(\frac{v - v_P}{2} \right) \sin \left(\frac{v - v_{P'}}{2} \right) \partial_v . \quad (41)$$

The advantage is that now the general nature of the flow of the Killing vector on \mathcal{J} is evident (figure 3). In particular the flow will be lightlike along the lightlike lines $u = u_P$ or $u_{P'}$, $v = v_P$ or $v_{P'}$. The parameter k is an additional parameter not determined by the location of the fixed points of the flow, and will—as we will see—enter into the formula for the spin of the BTZ black hole.

We can now perform a conformal rescaling on a region of \mathcal{J} where the flow is spacelike such that the norm of our chosen Killing vector becomes unity. To be precise about it, we choose

$$d\hat{s}^2 = \omega^{-1} d\hat{s}^2 , \quad (42)$$

where the conformal factor is

$$\omega = -k \sin \left(\frac{u - u_P}{2} \right) \sin \left(\frac{u - u_{P'}}{2} \right) \sin \left(\frac{v - v_P}{2} \right) \sin \left(\frac{v - v_{P'}}{2} \right) . \quad (43)$$

When the identification is carried through the Killing vector ξ , whose norm on \mathcal{J} is now unity, will serve as the generator of rotations. There is an orthogonal

timelike Killing vector with unit norm that will serve as the generator of time translations. The event horizon is the backwards light cone of the last point on \mathcal{J} , as usual. Fortunately it is not really necessary to go into spacetime to fetch its generator. It can be found by means of a calculation carried through entirely on \mathcal{J} , as the Killing vector field that is normal to the surface

$$\omega = 0 . \quad (44)$$

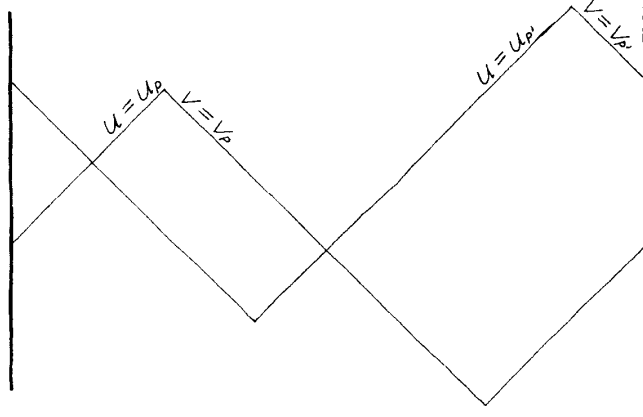


Fig. 3. This figure again is drawn on the surface of the anti-de Sitter cylinder. The null lines are the Killing horizons for a type I_b Killing field “in general position” (eqn. (41)). The flow is spacelike within the two rectangular regions and has fixed points where these lines intersect.

Finally we express the horizon generator as a linear combination of the generators of time translation and rotation, and read off the angular velocity of the event horizon relative to \mathcal{J} . To cut a fairly long story short, the result is

$$\Omega = \frac{\sin\left(\frac{u_P - u_{P'}}{2}\right) - k \sin\left(\frac{v_P - v_{P'}}{2}\right)}{\sin\left(\frac{u_P - u_{P'}}{2}\right) + k \sin\left(\frac{v_P - v_{P'}}{2}\right)} , \quad (45)$$

where (u_P, v_P) are the light cone coordinates of the last point on \mathcal{J} and $(u_{P'}, v_{P'})$ those of the first point. As in the special case considered earlier, the horizon generator will be the normal of a bifurcate Killing horizon that forms the inwards boundary of the domain of exterior communication of the black hole. This observation as well as the formula for Ω will prove useful below.

5. The Exterior of a Spinless Wormhole.

The next and final stage in our preparations for the spinning wormhole is to revisit the spinless wormhole. Again the task is to select some discrete subgroup Γ of isometries of anti-de Sitter space and then to take the quotient of a suitable open region of this space with Γ . The quotient space will necessarily have constant curvature; the difficult part is to ensure that its spatial topology is (say) a torus with a disk cut out, that \mathcal{J} exists in the quotient space and that the "singularities" in the interior are hidden by an event horizon. The wormhole will be a spinless black hole provided that all the elements of Γ belong to the diagonal subgroup $SO(2, 1)$ of $SO(2, 2)$ consisting of group elements that transform the Poincaré disk $V = 0$ onto itself, and we have to arrange matters so that only hyperbolic group elements occur. In our previous publication [3] we solved the entire problem through the specification of a fundamental region that defines Γ . It turns out that this approach is impracticable in the spinning case, and therefore we will now present the spinless wormhole in covering space. This is enough to discuss the causal structure of the solution.

We will concentrate on a simple and quite symmetric example, leaving the general case to take care of itself. Thus we choose the type I_b group elements

$$\gamma_1 = e^{\xi_1} , \quad \xi_1 = aJ_{XU} \quad (46)$$

$$\gamma_2 = e^{\xi_2} , \quad \xi_2 = aJ_{YU} . \quad (47)$$

We assume that $a > 0$. The second Killing vector is obtained through a 90 degree rotation of the first. Now we take our discrete group Γ to be the free group that is formed by all possible "words" constructed from these generators. This group has an infinite set of elements, and it would be a very hard task to obtain an explicit characterization of all of them. In order to understand the action of Γ on anti-de Sitter space it will therefore be necessary to proceed somewhat indirectly.

The main point at issue is to find conditions on the parameter a such that all the elements of Γ can be generated by type I_b Killing vectors. The generators of Γ are realized as hyperbolic Möbius transformations of the Poincaré disk. What we have to know about hyperbolic Möbius transformations is that they each have two fixed points on the boundary of the disk, and that there is one and only one flow line of a hyperbolic Möbius transformation that is also a geodesic connecting the fixed points. (Elliptic Möbius transformations are not allowed in this context precisely because they always have a fixed point within the disk, and this would give rise to a conical singularity in the quotient space.) All that we have to do to ensure that Γ consists of hyperbolic Möbius transformations only is to choose a fundamental region such that it tessellates the disk and becomes a smooth manifold when its sides are identified—the identification of

a pair of sides then defines the action of the generators of the group. Figure 4 should make it clear how we choose the fundamental region. The boundaries of this region must meet the boundary of the disk without crossing each other, and one may show that this condition is met whenever

$$\sinh^2 \frac{a}{2} > 1 . \quad (48)$$

If this condition is not met \mathcal{J} disappears and there is a conical singularity in the quotient space; in group theoretical terms the group Γ would contain elliptic elements.

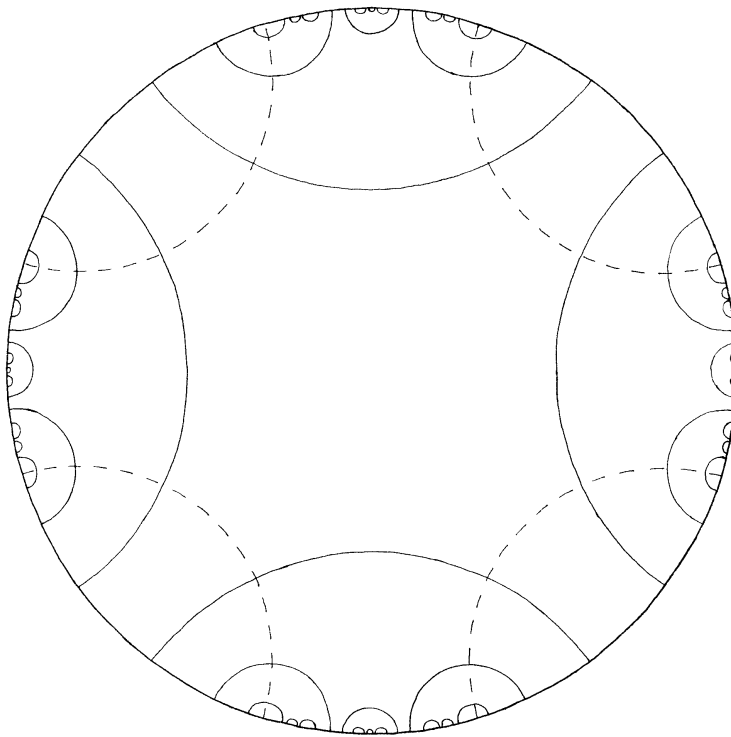


Fig. 4. An "initial data surface" for a time symmetric worm hole. It shows a fundamental region whose sides are identified by γ_1 and γ_2 , and how the disk is tessellated by copies of this region. The geodesic flow lines of some of the horizon words—namely (49) with permutations—are shown as dashed lines. In the quotient space they form the event horizon of the black hole.

The covering space of the moment of time symmetry ($V = 0$) is the entire disk and the entire boundary minus the fixed points. Every element of Γ contributes two fixed points, and every "word" in the generators is a group element.

An important example is

$$\gamma_{hor} = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} . \quad (49)$$

We call this word a horizon word, since—as we will explain below—the unique geodesic connecting its fixed points lies in the event horizon of the black hole. The group Γ contains an infinite set of copies of this word obtained by means of conjugation by arbitrary group elements γ :

$$\gamma^{-1} \gamma_{hor} \gamma = \gamma^{-1} \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} \gamma . \quad (50)$$

All these words, which include the permutations of the group element (49), will also be called horizon words. The Killing fields that generate them are all representatives of one and the same field in the quotient space, namely, the generator of rotations for the wormhole.

We need a good qualitative understanding of the set of all fixed points, or rather, the set of fixed point *free* segments of \mathcal{J} at $t = 0$. In order to obtain this set, begin by drawing the fundamental region in the disk. The four segments of the boundary of the disk that belong to the fundamental region obviously are free from fixed points. Next draw all copies of the fundamental region that can be obtained by transforming it with γ_1 , γ_2 and their inverses. In this way a number of new fixed point free segments appear as copies of the first ones. Namely, the four fixed point free segments that we started out with are prolonged, and eight additional such segments appear. Continue in this way, i.e. add further copies of the fundamental region until the entire disk has been tessellated, and in each step enlarge the set with all the copies of the fixed point free segments obtained in the preceding step. Note that this construction is reminiscent of the way in which the Cantor dust is obtained, only that in this case we are interested in the complement to the dust, rather than in the dust itself. Two key observations are, first, that the original fixed point free segments will grow so that they become segments ending in the fixed points of the horizon words, and, second, that every other fixed point free segment is situated between the fixed points of some horizon word. The upshot of this discussion is that the covering space of the boundary at $t = 0$ can be described as the union of the fixed point free segments associated to the infinite set of horizon words.

Having understood the "initial data" surface it is easy to understand the covering space of the \mathcal{J} of the spinless wormhole. By definition this covering space is the intersection of all the regions where the flow of some element of Γ is spacelike (all the flows are spacelike in the intersection), but a more useful description is to say that it is the union of a set of squares in one to one correspondence with all horizon word copies. Let us show this. We have a qualitative understanding of the location of the fixed points on the line $t = 0$.

Each element in the group Γ is associated with a unique pair of such fixed points. Moreover each such fixed point is the vertex of two lightlike lines along which the Killing vector generating this particular element of Γ becomes lightlike. When we draw all these lines (in practice: some of them) we find that the boundary of the original anti-de Sitter space becomes divided into squares. Each fixed point free segment of the line $t = 0$ becomes the diagonal of a square in which the flow of every element of Γ is spacelike. Hence the covering space of the \mathcal{J} of the wormhole is given by the union of all such squares. But by the previous paragraph the fixed point free segments are in one to one correspondence with the copies of the horizon word, so this is what we wanted to show. There are four large squares lying between the fixed points of the original horizon words (that is, (49) with permutations) and the future and past corners of these squares define the last and first points on \mathcal{J} , respectively, see figure 5. It will be useful to quote the formula for the location of the last point on \mathcal{J} . It occurs [3] at $(t, \phi) = (t_P, \phi_P)$, where

$$\tan t_P = \sqrt{2 \tanh^2 \frac{a}{2} - 1} \quad \phi_P = \frac{\pi}{4} + \frac{n\pi}{2} . \quad (51)$$

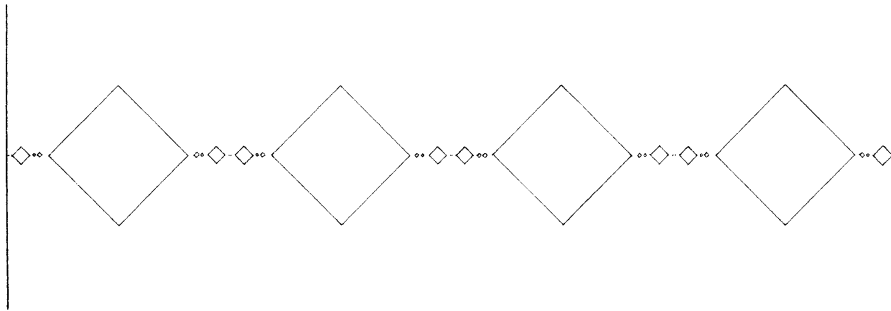


Fig. 5. This infinite set of squares on \mathcal{J} (of which only a few are large enough to be visible in this figure) is in one to one correspondence with all copies of the horizon word. Namely, the corners on $t = 0$ of one of the squares are the two fixed points of precisely one horizon word. Since there are no fixed points of Γ within the squares, all our Killing flows are spacelike there. Thus the union of their interiors is the covering space of \mathcal{J} for the spinless wormhole.

More precisely there are four such points situated at 90 degrees distance around the boundary; the first point on \mathcal{J} has the sign of its t -coordinate reversed. Note that, appearances notwithstanding, the squares are isometric copies of each other—indeed we choose the scale of the metric at \mathcal{J} so that this statement is true.

We will defer a discussion of the interior of the wormhole spacetime till section 7. However, let us give a few brief remarks that suffice to understand its domain of exterior communication. The basic point is that each of the squares on \mathcal{J} is, in itself, simply an instance of the covering space of one asymptotic region of a BTZ black hole "in arbitrary position", as discussed in the previous section. By definition the covering space of the event horizon is the backwards light cone of the last point on \mathcal{J} ; since this is a totally geodesic surface its intersection with the totally geodesic surface $t = 0$ is itself a geodesic, and it becomes a closed geodesic when the identification is carried through. Now we already know a closed geodesic in that homotopy class, namely the unique geodesic flow line of the generator of the horizon word, and therefore that flow line does indeed lie in the event horizon. There is another insight that comes for free at this point: Every square defines a bifurcate Killing horizon in the interior, defined by the backwards light cone from its future corner and the forwards lightcone from its past corner. The region in between \mathcal{J} and this bifurcate Killing horizon is indeed isometric to the covering space of the domain of exterior communication of the BTZ black hole. This will remain true of the quotient spaces; hence the conclusion is that the domain of exterior communication of the time symmetric wormhole is isometric to that of the spinless BTZ black hole. Note that this does not [3] mean that the non-trivial topology in the interior is unobservable from \mathcal{J} .

6. The Exterior of a Spinning Wormhole.

Now we turn to the main point of this paper, the construction of a spinning wormhole. We concentrate on a simple and symmetric example where the Killing vectors are related by a rotation through 90 degrees. Thus we choose the type I_b group elements

$$\gamma_1 = e^{\xi_1}, \quad \xi_1 = aJ_{XU} + bJ_{YV} \quad (52)$$

$$\gamma_2 = e^{\xi_2}, \quad \xi_2 = aJ_{YU} - bJ_{XV}. \quad (53)$$

We assume that $a > b \geq 0$. The group Γ is defined as the free group formed from these generators. At first sight it may seem that we must now understand Γ more or less "in the raw", since it will no longer be true that there is some particular Poincaré disk in anti-de Sitter space that is transformed into itself by all the elements of Γ . Nevertheless we will see that the general case is only marginally more difficult than the time symmetric case that we studied in the previous section. Let us first write the generating Killing vectors so that it becomes manifest how they act on \mathcal{J} :

$$\xi_1 = -(a+b)J_1 - (a-b)\tilde{J}_1 = -(a+b)\sin u\partial_u - (a-b)\sin v\partial_v \quad (54)$$

$$\xi_2 = (a+b)J_2 - (a-b)\tilde{J}_2 = -(a+b)\cos u\partial_u + (a-b)\cos v\partial_v . \quad (55)$$

The point to notice is that when we exponentiate the action of these generators then the transformation of the u coordinate will be determined by the parameter $(a+b)$, while that of the v coordinate will be determined by $(a-b)$. We may as well do this at once. The generators of Γ act as projective transformations on the circles that are coordinatized by u and v . It is convenient to introduce "rectifying" coordinates

$$z = \tan \frac{u}{2} \quad \tilde{z} = \tan \frac{v}{2} . \quad (56)$$

Then we find after a short calculation that

$$\gamma_1 = e^{\xi_1} : \quad z \rightarrow z' = \frac{e^{-\frac{a+b}{2}z}}{e^{\frac{a+b}{2}z}} \quad (57)$$

$$\gamma_2 = e^{\xi_2} : \quad z \rightarrow z' = \frac{\cosh \frac{a+b}{2}z - \sinh \frac{a+b}{2}z}{-\sinh \frac{a+b}{2}z + \cosh \frac{a+b}{2}z} , \quad (58)$$

and similarly for the coordinate \tilde{z} but with the parameter $(a+b)$ replaced with $(a-b)$. This is the action of the generators of the group Γ . To ensure that \mathcal{J} exists we have to choose the parameters so that all the elements of Γ belong to the conjugacy class $I_b = \text{hyp} \otimes \text{hyp}$. But we now see that this can be ensured by choosing the parameter $(a+b)$ so that all the projective transformations of the circle coordinatized by u or z are hyperbolic, and at the same time choosing the parameter $(a-b)$ so that the same is true for the circle coordinatized by v or \tilde{z} . All that has happened is that we have two copies of the problem that we solved in the time symmetric case, and it follows that the conditions that ensure that \mathcal{J} exists are

$$\sinh^2 \frac{a+b}{2} > 1 \quad \sinh^2 \frac{a-b}{2} > 1 . \quad (59)$$

It remains to elucidate the properties of the resulting quotient space.

A qualitative understanding of \mathcal{J} is in fact already in hand; on the line $t = 0$ we mark the fixed points of a spinless wormhole with parameter $(a+b)$ and draw lines of constant u through these points. Then we change the parameter to $(a-b)$, mark the fixed points, and draw lines of constant v through those. The fixed points of the spinning wormhole will occur where two such lines belonging to the same group element meet. In the spinless case the set of fixed points lying on the line $t = 0$ played a special role. As we increase the parameter b these fixed points move away from this particular line, but it is possible to show that they still lie on a spacelike (but quite "jagged") line.

Indeed the whole picture is very similar to what it was in the spinless case, the only difference is that what used to be squares where the flow of all the

Killing vectors is spacelike has become rectangles, as shown in figure 6. The rectangles are isometric copies of each other and each rectangle can be regarded as an instance of the covering space of \mathcal{J} for a BTZ black hole "in general position". Moreover each rectangle defines a bifurcate Killing horizon in the interior and the domain of exterior communication of our wormhole is covered by the union of all such domains, one for each rectangle. Hence it is isometric to the domain of exterior communication for a single BTZ black hole. The event horizon is given by the backwards lightcone of the last point on \mathcal{J} , as usual. When we go to the quotient space only one asymptotic region will remain (as can be shown by choosing a fundamental region on \mathcal{J}).

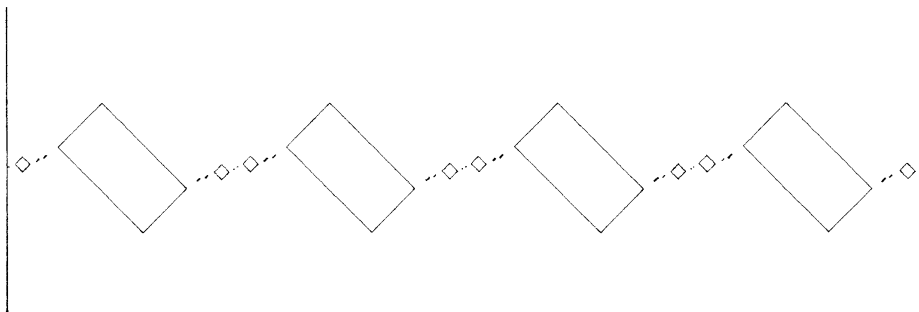


Fig. 6. This is the analogue of figure 5 for the spinning case: the union of the rectangles shown is the covering space of \mathcal{J} for the spinning wormhole. The corners of the large rectangles are the fixed points of the horizon word (49) with permutations, and the first and last points on \mathcal{J} . The fixed points that in the spinless case occurred on the line $t = 0$ now lie on spacelike (but quite "jagged") lines connecting the fixed points of all the horizon words.

We want to compute the angular velocity of the horizon with respect to \mathcal{J} . This is a considerably messy calculation. In logical outline what we have to do is to compute the generator of rotation from the equation

$$e^{\xi_{rot}} = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} . \quad (60)$$

Then we compare the result to our previous general form of a type I_b Killing vector, eq. (39), and read off the parameter k . This parameter together with the known values for the light cone coordinates of the fixed points that terminate \mathcal{J} to the future and past can then be inserted into our general formula for the angular velocity, eq. (45). We were unable to simplify this calculation very much. Anyway, the result is

$$\Omega = \frac{c - \tilde{c}}{c + \tilde{c}} , \quad (61)$$

where

$$c = -2 \sinh^2 \left(\frac{a+b}{2} \right) \sqrt{\sinh^4 \left(\frac{a+b}{2} \right) - 1} \quad (62)$$

and

$$\tilde{c} = -2 \sinh^2 \left(\frac{a-b}{2} \right) \sqrt{\sinh^4 \left(\frac{a-b}{2} \right) - 1} . \quad (63)$$

These formulæ provide the desired quantitative expression for the rate of spin of our wormhole.

It remains to count the number of parameters in our solution. To do so we observe that six parameters are needed to characterize an element of the six dimensional isometry group $SO(2, 2)$. The discrete group Γ is generated by two such elements, and we must remember that the final spacetime is invariant under a global isometry described by six parameters. Hence the number of parameters in our solution is $2 \times 6 - 6 = 6$. In the time symmetric case the group elements are to be chosen in the three dimensional group $SO(2, 1)$, so that the number of parameters in the time symmetric case is only $2 \times 3 - 3 = 3$. In effect we are using the analogue of Fricke-Klein coordinates on the parameter space, while in our previous publication [3] we used Fenchel-Nielsen coordinates. As we will see in the next section the spatial topology of our wormhole is that of a torus with an asymptotic region attached. The generalization to the higher genus case is straightforward in principle.

As an aside, we note that it is rather difficult to understand the parameter spaces in terms of the kind of presentation of the group that we have been employing here. Consider the time symmetric case for simplicity. The three parameters are the parameters a and a' multiplying the Killing vectors that are to be exponentiated to get the generators, and the angle α between their geodesic flow lines in the initial data disk. The condition on these parameters that guarantee that a black hole solution results is

$$\coth^2 \frac{a}{2} + \coth^2 \frac{a'}{2} - \coth^2 \frac{a}{2} \coth^2 \frac{a'}{2} > \cos^2 \alpha . \quad (64)$$

When $a = a'$ and $\alpha = \pi/2$ this reduces to eq. (48).

7. The Interior of the Spinning Wormhole.

It is time to discuss the interior of our spinning wormhole. It is *a priori* not so clear whether—like the spinning BTZ black hole as well as the ordinary Kerr solution—our spinning wormhole has a mouth behind the event horizon through which an observer may travel into a new universe, identical with the one he left. As a matter of fact this is not the case. To see this we have to understand the covering space of the interior.

First recall the situation for the non-rotating wormhole. In that case all the elements of Γ act as hyperbolic Möbius transformations in the Poincaré disk at $t = 0$. In particular they all have two fixed points at its boundary. As may be shown, the "singularity surface" belonging to such an element consists of lightlike surfaces growing up forwards and backwards in time from each of its two fixed points at $t = 0$. In other words, they are light cones with their apices at \mathcal{J} . The covering space—by definition a region where all the flows are spacelike—is then easily visualized. Just locate the fixed points at $t = 0$ of all the elements in Γ , and remove all points in the causal past and the causal future of these fixed points. What remains is the covering space, which we have tried to depict in figure 7. It lasts only for a finite amount of coordinate time, that is until all these light cones meet in the interior. In our coordinates this will happen at $t = \pm\pi/2$ and $\rho = 0$.

In the spinning case things get more involved because the "singularity surfaces" are timelike. In the case of the spinning BTZ black hole this has the consequence that a mouth opens up in the disk at $t = \pi/2$ through which an observer may travel to another universe, identical with the one she left. We will show that this does not happen in the spinning wormhole. Instead it turns out that there are certain elements of the group Γ whose "singularity surfaces" get arbitrarily close to null planes, i.e. lightcones with their apices at infinity. Moreover there is such an apex arbitrarily close to any given fixed point on the jagged spacelike line where—according to section 6—the fixed points lie. Thus the covering space can be visualized in much the same way as in the non-rotating case, that is as what is left when all points to the future and to the past of these fixed points have been removed.

Since we will consider families of timelike surfaces that have a null surface as a limit we should make clear that this notion makes sense because we regard the coordinate system as fixed in the discussion. The idea is that all points on one side of any surface in the family are to be removed from covering space (since the Killing flow is timelike there). If it is the case that any given point on one side of a null surface is such that a member of the family of timelike surfaces passes between the point and the null surface, then the family is said to have the null surface as a limit, and the given point must be removed from covering space. In effect the null surface can be regarded as a "singularity surface", except that points on the null surface itself are not removed from covering space.

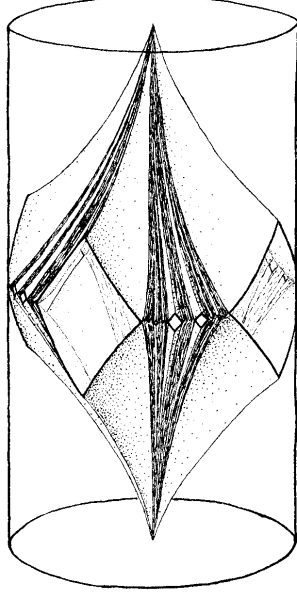


Fig. 7. Here we have tried to visualize the full covering space of the spinless wormhole in the anti-de Sitter cylinder. Thus figure 5, which shows its covering space on \mathcal{I} only, is simply the surface of this cylinder. Indeed, two of the four large squares of that figure are visible also here, while the other two are hidden behind the structure. Formally, to obtain the boundary of the full covering space from figure 5 we should evolve lightcones forward and backward from the fixpoints at $t = 0$ of all the elements in Γ . Since it takes a time $\pi/2$ for a lightray to travel from infinity to the centre of the cylinder, the resulting diamond shaped structure will last between $t = -\pi/2$ to $t = +\pi/2$.

Let us first show how this works for the fixed points of the generator γ_2 . Consider the group element

$$\gamma_m \equiv \gamma_2^m \gamma_1 \gamma_2^{-m} . \quad (65)$$

This is simply the generator γ_1 transformed m times by γ_2 . Somewhat lengthy calculations shows that this group element is obtained by exponentiating the Killing vector

$$\begin{aligned} \xi_m = & -(a+b)(J_1 \cosh m(a+b) + J_3 \sinh m(a+b)) - \\ & -(a-b)(\tilde{J}_1 \cosh m(a-b) - \tilde{J}_3 \sinh m(a-b)) . \end{aligned} \quad (66)$$

The norm of ξ_m becomes

$$\begin{aligned} \|\xi_m\|^2 = & \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{4} \left(e^{2ma}(Y - U)^2 + \right. \\ & \left. + e^{-2ma}(Y + U)^2 - e^{2mb}(X + V)^2 - e^{-2mb}(X - V)^2 \right) . \end{aligned} \quad (67)$$

Now let us take the limit $m \rightarrow \infty$. If $Y \neq U$ we get (since $a > b$)

$$\lim_{m \rightarrow \infty} \|\xi_m\|^2 = \frac{a^2 - b^2}{4} e^{2ma}(Y - U)^2 . \quad (68)$$

The "singularity surface" of the element γ_m is defined as the surface where ξ_m is null, and it coincides with the "singularity surface" of γ_1 transformed m times by γ_2 . What we have shown is that in the limit $m \rightarrow \infty$ this surface approaches the surface $Y = U$, which is a light cone with its vertex at one of the fixed points of γ_2 . Note however that eq. (68) fails on this surface; if we set $Y = U$ before we take the limit we see that the ξ_m is actually a timelike vector on this surface.

We can already conclude that there can be no mouth inside the spinning wormhole, because the "singularity surfaces" of γ_m and its inverse for large enough m get arbitrarily close to lightcones with apices at the fixed points of γ_2 . These meet each other at $t = \pi/2$, so if the covering space does not end before that it certainly does so then.

We would also like to show that there are no inner horizons. To do this we have to show that the boundary of covering space is everywhere lightlike rather than timelike, that is to say that we have to show that there is a "singularity surface" that is arbitrarily close to a light cone having its apex at an arbitrary given fixed point. We will offer a heuristic argument to this effect. Consider the "singularity surface" of the word

$$\eta^m \gamma \eta^{-m} , \quad (69)$$

where η and γ are arbitrary elements of Γ . The claim is that for large m this surface approaches a light cone with its apex at one of the fixed points of η .

Let $\eta = e^\xi$ and let S_m be the timelike "singularity surface" of γ transformed m times with η . We assume that ξ is never tangent to S_0 , so that indeed the entire surface is transformed by η . That is, we assume that

$$\frac{\xi}{\|\xi\|} \cdot n_0 \neq 0 , \quad (70)$$

where n_0 is the normal of S_0 . But the normal n_m to the surface S_m is defined as the transformation of n_0 along the Killing flow of ξ , and therefore it must be true that

$$\frac{\xi}{\|\xi\|} \cdot n_m = \frac{\xi}{\|\xi\|} \cdot n_0 . \quad (71)$$

Now we want to know what happens to S_m as m goes to infinity. We assume that it becomes a limiting surface S_∞ . But since ξ has a component along the normal of the surfaces for all m the limiting surface can exist only if the limiting normal lies in the limiting surface, that is to say that S_∞ must be a null surface. Since its apex on \mathcal{J} is being pushed by η towards one of its own fixed points it follows that for large enough m the "singularity surface" S_m will come arbitrarily close to a light cone with its apex at one of the fixed points of the arbitrary group element η . This proves our claim, provided that the assumptions that we made are accepted. Hence the covering space of the spinning wormhole can be visualized in the same manner as the covering space of the spinless wormhole, and there can be no inner horizons.

Finally, the topology of the quotient space: Is it a wormhole? Yes. According to the previous section there is only one asymptotic region attached. Moreover the first homotopy group of the quotient space is equal to Γ , just as it was in the case of the spinless wormhole (even though the details of how Γ acts on the covering space have been changed). Hence the topology of the quotient space is the same in both cases, namely that of a torus with an asymptotic region attached. Note that higher genus wormholes can be constructed if they are wanted.

8. Conclusions and Open Questions.

Our conclusions are simple to state. We have constructed a spacetime of the form adS/Γ whose spatial topology is that of a torus with an asymptotic region attached. The domain of exterior communication is isometric to that of the spinning BTZ black hole [8], which means that there is an event horizon that becomes stationary at late times and that spins with respect to infinity. All "singularities" in the interior are hidden behind this event horizon. The interior ends in a "singularity" and there are no inner horizons. The solution is a member of a six parameter family of solutions having similar properties (and which includes a three parameter subfamily of spinless wormholes).

Our argument for the absence of inner horizon was heuristic and ought to be tightened up. This should be doable. To see what the open questions are, let us remind ourselves about the spinless wormholes [3]. The parameter space of all such solutions is well understood for arbitrary genus. Their domain of exterior communication is isometric to that of the spinless BTZ black hole. We also have a detailed description of the topology of the event horizon (including its caustics), the location of all the marginal trapped surfaces is known, and it is understood in what sense active—but not passive—topological censorship holds. Clearly it would be of interest to bring our understanding of the spinning

case up to the same level, so for the spinning case this becomes a list of open question; we expect that they should prove easy to settle. A different question that should be addressed at the classical level is the dynamics of these spacetimes from a Hamiltonian point of view. The case of compact locally anti-de Sitter spacetimes is well understood [14]; the non-compact case considered here offers complications of some interest.

In spite of the loose ends we think that it is fair to say that the Clifford-Klein problem that we have posed is solved. There is a similar problem in 3+1 dimensions: Find all suitably regular 3+1 dimensional spacetimes of the form adS/Γ . Natural generalizations of the BTZ black holes do exist [15]. These solutions have been explored and extended in various directions [9] [16], but it is conceivable that further solutions of interest may exist.

Beyond these classical questions there looms the question whether our worm-holes can provide soluble examples that may help us to understand the quantum mechanics of black holes and topological geons in general. Whether this is so remains to be seen.

Acknowledgement.

Ingemar Bengtsson was supported by NFR.

References

- [1] C.W. Misner and J.A. Wheeler, Ann. Phys. 2 (1957) 525.
D.R. Brill and R. Lindquist, Phys. Rev. 131 (1963) 471.
- [2] G.W. Gibbons, Commun. Math. Phys. 27 (1972) 87.
D. Gannon, J. Math. Phys. 16 (1975) 2364.
- [3] S. Åminneborg, I. Bengtsson, D.R. Brill, S. Holst and P. Peldán, Class. Quant. Grav. 15 (1998) 627.
- [4] G. Mess, Lorentz Spacetimes of Constant Curvature, IHES preprint, April 1990.
- [5] W. Killing: Einführung in die Grundlagen der Geometrie I, Paderborn 1893.
- [6] J.L. Friedman, K. Schleich and D.M. Witt, Phys. Rev. Lett. 71 (1993) 1486.
- [7] S. Carlip, Phys. Rev. D51 (1995) 622.

- A. Strominger, Black Hole Entropy from Near-Horizon Microstates, hep-th/9712251.
- [8] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849.
M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D48 (1993) 1506.
- [9] S. Holst and P. Peldán, Class. Quant. Grav. 14 (1997) 3433.
- [10] D. Brill, Phys. Rev. D53 (1996) R4133.
A. Steif, Phys. Rev. D53 (1996) 5527.
- [11] R.M. Wald, General Relativity, University of Chicago Press, Chicago 1984.
- [12] A. Ashtekar and A. Magnon, Class. Quant. Grav. 1 (1984) L39.
- [13] J.D. Brown and M. Henneaux, Commun. Math. Phys. 104 (1986) 207.
- [14] L. Andersson, V. Moncrief and A.J. Tromba, J. Geom. Phys. 23 (1997) 191.
- [15] S. Åminneborg, I. Bengtsson, S. Holst and P. Peldán, Class. Quant. Grav. 13 (1996) 2707.
D.R. Brill and J. Louko, unpublished.
- [16] M. Bañados, Phys. Rev. D57 (1998) 1068.
D.R. Brill, J. Louko and P. Peldán, Phys. Rev. D56 (1997) 3600.
D. Klemm, V. Moretti and L. Vanzo, Rotating Topological Black Holes, gr-qc/9710123.
R. B. Mann and W.L. Smith, Phys. Rev. D56 (1997) 4942.